Damping to prevent the blow-up of the Korteweg-de Vries equation

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Abstract. We study the behavior of the solution of a generalized damped KdV equation $u_t + u_x + u_{xxx} + u^p u_x + \mathscr{L}_\gamma(u) = 0$. We first state results on the local well-posedness. Then when $p \geq 4$, conditions on $\mathscr{L}_\gamma$ are given to prevent the blow-up of the solution. Finally, we numerically build such sequences of damping.

Keywords. KdV equation, dispersion, dissipation, blow-up.

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Introduction

The Korteweg-de Vries (KdV) equation is a model of one-way propagation of small amplitude, long wave [KdV95]. It is written as

$$u_t + u_x + u_{xxx} + uu_x = 0.$$

In [BDKM96], Bona et al. consider the initial- and periodic-boundary-value problem for the generalized Korteweg-de-Vries equation

$$u_t + u_{xxx} + u^p u_x = 0$$

and study the effect of a dissipative term on the global well-posedness of the solution. Actually, they consider two different dissipative terms, a Burgers-type one $-\delta u_{xx}$ and a zeroth-order term σu. For both these terms, they show that for $p \geq 4$, there exist critical values δ_c and σ_c such that if $\delta > \delta_c$ or $\sigma > \sigma_c$ the solution is globally well-defined. However, the solution blows-up when the damping is too weak as for the KdV equation [MM02].The literature is full of work concerning the dampen KdV equation with $p = 1$ [ABS89, CR04, CS13b, CS13a, Ghi88, Ghi94, Gou00, GR02], but few are concerning more general nonlinearities.

In our paper, we consider a more general damping term denoted by $\mathscr{L}_\gamma(u)$. Our purpose is to find similar results as above, both theorically and numerically. So the KdV equation becomes a damped KdV (dKdV) equation and is written

$$u_t + u_x + u_{xxx} + u^p u_x + \mathscr{L}_\gamma(u) = 0.$$

The damping operator $\mathscr{L}_\gamma(u)$ works on the frequencies. It is defined by its Fourier symbol

$$\widehat{\mathscr{L}_\gamma(u)}(\xi) := \gamma(\xi)\hat{u}(\xi).$$

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Here $\hat{u}$ is the Fourier transform of u and γ a strictly positive function chosen such that

$$\int_{\mathbb{R}} u(x)\mathscr{L}_\gamma(u)d\mu(x) = \int_{\mathbb{R}} \gamma(\xi)|\hat{u}(\xi)|^2 d\xi \geq 0.$$

We notice than the two cases studied in [BDKM96] are present with this damping by taking $\gamma(\xi) = \delta\xi^2$ and $\gamma(\xi) = \sigma$ respectively.

The KdV equation has an infinite number of invariants such that the L^2-norm. But, for the dKdV equation, the L^2-norm decreases. Indeed, for all $t \in \mathbb{R}$,

$$\frac{d}{dt}\|u\|_{L^2}^2 = -|u|_\gamma^2$$

where the natural space of study is

$$H_\gamma(\mathbb{R}) := \left\{ u \in L^2(\mathbb{R}) \text{ s.t. } \int_{\mathbb{R}} \gamma(\xi)\,|\hat{u}(\xi)|^2\, d\xi < +\infty \right\}$$

and the associated norm is

$$|u|_\gamma := \sqrt{\int_{\mathbb{R}} \gamma(\xi)\,|\hat{u}(\xi)|^2\, d\xi}.$$

An other property of the KdV equation is that the solution can blow-up as soon as $p \geq 4$. The blow-up is caracterized by $\lim\limits_{t\to T} \|u\|_{H^1} = +\infty$.

In this paper, we first establish the local well-posedness of the dKdV equation. Then we study the global well-posedness. More precisely, we focus on the behavior of the H^1-norm with respect to p and we obtain conditions on γ so there is no blow-up. Finally, we illustrate the results using some numerical simulations. We first find a constant damping ($\gamma(\xi)$ =constant) such that there is no blow-up and then the damping is weaken in such a way $\lim\limits_{|\xi|\to+\infty} \gamma(\xi) = 0$.

1 Preliminary results

Some results of injection concerning the space $H_\gamma(\mathbb{R})$ are given.

Proposition 1.1. *Assume $\int_{\mathbb{R}} \frac{1}{\gamma(\xi)} < +\infty$ then there exists a constant $C > 0$ such that $\|u\|_\infty \leq C\,|u|_\gamma$, i.e., the injection $H_\gamma(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous.*

Proof. Let $u \in H_\gamma(\mathbb{R})$. We notice that

$$u(x) = \int_{\mathbb{R}} \hat{u}(\xi)e^{i\xi x} d\xi.$$

Then

$$|u(x)| \leq \int_{\mathbb{R}} |\hat{u}(\xi)| = \int_{\mathbb{R}} \frac{1}{\sqrt{\gamma(\xi)}}\sqrt{\gamma(\xi)}|\hat{u}(\xi)|.$$

We assumed that $\gamma(\xi) > 0$. Hence, the Cauchy-Schwarz inequality involves for all $x \in \mathbb{R}$:

$$|u(x)| \leq \left(\int_{\mathbb{R}} \frac{1}{\gamma(\xi)}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \gamma(\xi)|\hat{u}(\xi)|^2\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}} \frac{1}{\gamma(\xi)}\right)^{\frac{1}{2}} |u|_\gamma\,.$$

□

Proposition 1.2. *Let γ and β be such that for all $\xi \in \mathbb{R}$, $\gamma(\xi) > \beta(\xi)$. We define*

$$\rho(N) := \max_{\xi \geq N} \frac{\beta(\xi)}{\gamma(\xi)}.$$

The continuous injection $H_\gamma(\mathbb{R}) \hookrightarrow H_\beta(\mathbb{R})$ is compact if and only if $\lim\limits_{N\to+\infty} \rho(N) = 0$.

Proof. The condition is necessary. Indeed, if there exists $\alpha > 0$ such that $\rho(N) > \alpha$, $\forall N$, then the norms $|u|_\beta$ and $|u|_\gamma$ are equivalent, the injection cannot be compact. Let us prove now that the condition is sufficient. First, we have for $u \in H_\gamma(\mathbb{R})$:

$$|u|_\beta = \int_{\mathbb{R}} \beta(\xi)|\hat{u}(\xi)|^2 \leq \int_{\mathbb{R}} \gamma(\xi)|\hat{u}(\xi)|^2 = |u|_\gamma .$$

This shows that the injection is continuous. Now we prove that the injection is compact. We use finite rank operators and we take the limit. Let I_N be the orthogonal operator on the polynomials of frequencies ξ such that $-N \leq \xi \leq N$. We have

$$I_N u = \int_{|\xi|\leq N} \hat{u}(\xi) e^{i\xi x} d\xi.$$

Thus

$$\begin{aligned} |(Id - I_N)u|_\beta^2 &= \int_{|\xi|>N} \beta(\xi)|\hat{u}(\xi)|^2, \\ &\leq \int_{|\xi|>N} \frac{\beta(\xi)}{\gamma(\xi)}\gamma(\xi)|\hat{u}(\xi)|^2, \\ &\leq \rho(N)\,|u|_\gamma^2 \underset{N\to+\infty}{\longrightarrow} 0. \end{aligned}$$

Therefore Id is a compact operator and consequently the injection is compact. $\square$

Proposition 1.3. *Assume that $u,\ v \in H_\gamma(\mathbb{R})$ and there exists a constant $C > 0$ such that $\forall \xi,\ \eta \in \mathbb{R}$ we have*

$$\sqrt{\gamma(\xi)} \leq C\left(\sqrt{\gamma(\xi-\eta)} + \sqrt{\gamma(\eta)}\right).$$

Then we have

$$|uv|_\gamma \leq C\left(|u|_\gamma \|\hat{v}\|_{L^1} + |v|_\gamma \|\hat{u}\|_{L^1}\right).$$

Moreover if $\int_{\mathbb{R}} \frac{1}{\gamma(\xi)} < +\infty$ then $H_\gamma(\mathbb{R})$ is an algebra.

Proof. Let $u,\ v \in H_\gamma(\mathbb{R})$. We have

$$|uv|_\gamma^2 = \int_{\mathbb{R}} \gamma(\xi)|\widehat{uv}(\xi)|^2.$$

We remind that $\widehat{uv}(\xi) = \hat{u} * \hat{v}(\xi)$. Using the inequality

$$\sqrt{\gamma(\xi)} \leq C\left(\sqrt{\gamma(\xi-\eta)} + \sqrt{\gamma(\eta)}\right),$$

we obtain for all $\xi,\ \eta \in \mathbb{R}$

$$\sqrt{\gamma(\xi)}|\widehat{uv}(\xi)| \leq C\left(\int_{\mathbb{R}} \sqrt{\gamma(\xi-\eta)}|\hat{u}(\xi-\eta)\hat{v}(\eta)|d\eta + \int_{\mathbb{R}} \sqrt{\gamma(\eta)}|\hat{u}(\xi-\eta)\hat{v}(\eta)|d\eta\right).$$

Hence

$$\begin{aligned} |uv|_\gamma^2 &\leq C^2 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \sqrt{\gamma(\xi-\eta)}|\hat{u}(\xi-\eta)\hat{v}(\eta)|d\eta + \int_{\mathbb{R}} \sqrt{\gamma(\eta)}|\hat{u}(\xi-\eta)\hat{v}(\eta)|d\eta\right)^2 d\xi, \\ &\leq C^2 \int_{\mathbb{R}} \left[\left(\int_{\mathbb{R}} \sqrt{\gamma(\xi-\eta)}|\hat{u}(\xi-\eta)\hat{v}(\eta)|d\eta\right)^2 + \left(\int_{\mathbb{R}} \sqrt{\gamma(\eta)}|\hat{u}(\xi-\eta)\hat{v}(\eta)|d\eta\right)^2\right] d\xi, \\ &\leq C^2 \left(\left\|\left(\sqrt{\gamma(\xi)}|\hat{u}|\right) * |\hat{v}|\right\|_{L^2}^2 + \left\||\hat{u}| * \left(\sqrt{\gamma(\xi)}|\hat{v}|\right)\right\|_{L^2}^2\right). \end{aligned}$$

However, for $f \in L^1$ and $g \in L^2$, we have

$$\| |f| * |g| \|_{L^2}^2 \le \|g\|_{L^2}^2 \|f\|_{L^1}^2.$$

Thus

$$|uv|_\gamma^2 \le C \left(|u|_\gamma^2 \|\hat{v}\|_{L^1}^2 + |v|_\gamma^2 \|\hat{u}\|_{L^1}^2 \right).$$

From proposition 1.1, we know there exists a constant $c > 0$ such that $\|\hat{u}\|_{L^1} \le c|u|_\gamma$ if $\int_{\mathbb{R}} \frac{1}{\gamma(\xi)} < +\infty$. Then, there exists $\tilde{C} > 0$ such that

$$|uv|_\gamma \le \tilde{C} |u|_\gamma |v|_\gamma.$$

□

2 Local well-posedness

We study the following Cauchy problem : $\forall x \in \mathbb{R},\ \forall t > 0$,

$$\begin{cases} u_t + u_x + u_{xxx} + u^p u_x + \mathscr{L}_\gamma(u) = 0, & (1) \\ u(x, t = 0) = u_0(x). & (2) \end{cases}$$

The semi-group generated by the linear part is written as

$$S_t u := \int_{\mathbb{R}} e^{i\xi x} e^{i(\xi^3 - \xi)t - \gamma(\xi)t} \hat{u}(\xi) d\xi.$$

In the rest of the section, $f(u)$ denotes the non-linear part of the equation, i.e., $f(u) = u^p u_x$. We first state a result of regularization.

Lemma 2.1. *Assume that $s,\ r \in \mathbb{R}^+$. Then there exists a constant $C_r > 0$, depending only on r, such that $\forall u \in H_{\gamma^s}(\mathbb{R})$ and $\forall t > 0$ we have*

$$|S_t u|_{\gamma^{s+r}}^2 \le \frac{C_r}{t^r} |u|_{\gamma^s}^2.$$

Proof. Let $r \in \mathbb{R}^+$, $u \in H_{\gamma^s}(\mathbb{R})$ and $t > 0$. Then we have

$$\begin{aligned} |S_t u|_{\gamma^{s+r}}^2 &= \int_{\mathbb{R}} \gamma(\xi)^{s+r} \left| e^{-\gamma(\xi)t} \hat{u}(\xi) \right|^2 d\xi \\ &\le \sup_{\xi \in \mathbb{R}} \left(\gamma(\xi)^r e^{-2\gamma(\xi)t} \right) |u|_{\gamma^s}^2. \end{aligned}$$

But $\forall \xi \in \mathbb{R}$

$$\gamma(\xi)^r e^{-2\gamma(\xi)t} \le \frac{\left(\frac{r}{2}\right)^r e^{-r}}{t^r} = \frac{C_r}{t^r}.$$

Thus

$$|S_t u|_{\gamma^{s+r}}^2 \le \frac{C_r}{t^r} |u|_{\gamma^s}^2.$$

□

Theorem 2.2. *Assume that there exists $r \in]0, 2[$ and for all $\xi \in \mathbb{R}$, $\gamma(\xi) \ge \xi^{\frac{2}{r}}$. We also assume that $\int_{\mathbb{R}} \frac{1}{\gamma(\xi)^s} < +\infty$ and there exists a constant $C > 0$ such that $\forall \xi,\ \eta \in \mathbb{R}$ and $s \in \mathbb{R}^+$ we have*

$$\sqrt{\gamma(\xi)^s} \le C \left(\sqrt{\gamma(\xi - \eta)^s} + \sqrt{\gamma(\eta)^s} \right).$$

Then there exists a unique solution in $\mathscr{C}([-T, T], H_{\gamma^s}(\mathbb{R}))$ of the Cauchy problem (1)-(2).

Moreover, for all $M > 0$ with $|u_0|_{\gamma^s} \le M$ and $|v_0|_{\gamma^s} \le M$, there exists a constant $C_1 > 0$ such that the solution u and v, associated with the initial data u_0 and v_0 respectively, satisfy for all $t \le \left(\frac{1}{C_0 M^p} \right)^{\frac{2}{r}}$

$$|u(\cdot, t) - v(\cdot, t)|_{\gamma^s} \le C_1 |u_0 - v_0|_{\gamma^s}.$$

Proof. Thanks to Duhamel's formula, $\Phi(u)$ is solution of the Cauchy problem, where

$$\Phi(u) = S_t u_0 - \int_0^t S_{t-\tau} f(u(\tau)) d\tau.$$

Let show that u is the unique fixed-point of Φ. We introduce the closed ball $\bar{B}(T)$ defined for $T > 0$ by

$$\bar{B}(T) := \left\{ u \in \mathscr{C}\left([0,T]; H_{\gamma^s}(\mathbb{R})\right) \text{ s.t. } |u(t) - u_0(t)|_{\gamma^s} \leq 3 |u_0|_{\gamma^s} \right\}.$$

We apply the Picard fixed-point theorem. We first show that $\Phi\left(\bar{B}(T)\right) \subset \bar{B}(T)$. Let us take $u \in \bar{B}(T)$ and show that $\Phi(u(t)) \in \bar{B}(T)$. We have

$$|\Phi(u(t))|_{\gamma^s} \leq |S_t u_0|_{\gamma^s} + \int_0^t |S_{t-\tau} f(u(\tau))|_{\gamma^s}.$$

On the one hand, we have

$$|S_t u_0|_{\gamma^s}^2 = \int_{\mathbb{R}} \gamma(\xi)^s \left|\widehat{S_t u_0}\right|^2 \leq \int_{\mathbb{R}} \gamma(\xi)^s |\hat{u_0}|^2 \leq |u_0|_{\gamma^s}^2.$$

On the other hand, we apply Lemma 2.1

$$\begin{aligned} |S_{t-\tau} f(u(\tau))|_{\gamma^s} &= |S_{t-\tau} f(u(\tau))|_{\gamma^{s-r+r}} \\ &\leq \frac{C_r}{(t-\tau)^{\frac{r}{2}}} |f(u(\tau))|_{\gamma^{s-r}}. \end{aligned}$$

But

$$\begin{aligned} |f(u(\tau))|_{\gamma^{s-r}}^2 &= \frac{1}{(p+1)^2} \int_{\mathbb{R}} \frac{\xi^2}{\gamma(\xi)^r} \gamma(\xi)^s \left|\widehat{u^{p+1}}\right|^2 d\xi \\ &\leq \frac{1}{(p+1)^2} \left|u^{p+1}\right|_{\gamma^s}^2 \end{aligned}$$

because $\gamma(\xi) > \xi^{\frac{2}{r}}$, and $H_{\gamma^s}(\mathbb{R})$ beeing an algebra, we have

$$|f(u(\tau))|_{\gamma^{s-r}} \leq C |u|_{\gamma^s}^{p+1}.$$

Consequently

$$\begin{aligned} |\Phi(u)|_{\gamma^s} &\leq |u_0|_{\gamma^s} + \int_0^t \frac{C}{(t-\tau)^{\frac{r}{2}}} |u|_{\gamma^s}^{p+1} d\tau \\ &\leq |u_0|_{\gamma^s} + C \sup_{t\in[0,T]} \left(|u|_{\gamma^s}^{p+1}\right) \int_0^t \frac{1}{(t-\tau)^{\frac{r}{2}}} d\tau \\ &\leq |u_0|_{\gamma^s} + \frac{C}{1-\frac{r}{2}} T^{1-\frac{r}{2}} \sup_{t\in[0,T]} \left(|u|_{\gamma^s}^{p+1}\right). \end{aligned}$$

But $u \in \bar{B}(T)$, then we have

$$|u(t)|_{\gamma^s} - |u_0|_{\gamma^s} \leq |u(t) - u_0|_{\gamma^s} \leq 3 |u_0|_{\gamma^s}.$$

That involves

$$|u(t)|_{\gamma^s} \leq 4 |u_0|_{\gamma^s},$$

and

$$\sup_{t\in[0,T]} \left(|u|_{\gamma^s}^{p+1}\right) \leq \left(\sup_{t\in[0,T]} |u|_{\gamma^s}\right)^{p+1} \leq 4^{p+1} |u_0|_{\gamma^s}^{p+1}.$$

We have $\Phi(u(t)) \in \bar{B}(T)$ if the inequality

$$|\Phi(u(t)) - u_0|_{\gamma^s} \leq 2\,|u_0|_{\gamma^s} + C_r T^{1-\frac{r}{2}} \left(4^{p+1}\,|u_0|_{\gamma^s}^{p+1}\right) \leq 3\,|u_0|_{\gamma^s}$$

is true i.e. if

$$0 < T^{1-\frac{r}{2}} \leq \frac{1}{4^{p+1} C_r\,|u_0|_{\gamma^s}^p}.$$

Now let us show that Φ is a strictly contracting map. Let $u,\ v \in \bar{B}(T)$, we prove that $\forall t \in [0,T]$,

$$\sup_{t\in[0,T]} |\Phi(u(t)) - \Phi(v(t))|_{\gamma^s} \leq k \sup_{t\in[0,T]} |u-v|_{\gamma^s}$$

with $k \in [0,1[$. As previously, we have

$$\begin{aligned}
|\Phi(u(t)) - \Phi(v(t))|_{\gamma^s} &= \left|\int_0^t S_{t-\tau}\left(f(u(\tau)) - f(v(\tau))\right) d\tau\right|_{\gamma^s} \\
&\leq \int_0^t \frac{C_0}{(t-\tau)^{\frac{r}{2}}} \left|u^{p+1} - v^{p+1}\right|_{\gamma^s}.
\end{aligned}$$

Using the equality

$$u^{p+1} - v^{p+1} = (u-v) \sum_{i+j=p} u^i v^j$$

and the injection results, we obtain

$$\begin{aligned}
\left|u^{p+1} - v^{p+1}\right|_{\gamma^s} &\leq C_1\,|u-v|_{\gamma^s} \left|\sum_{i+j=p} u^i v^j\right|_{\gamma^s} \\
&\leq C_2\,|u-v|_{\gamma^s} \sum_{i+j=p} |u|_{\gamma^s}^i\,|v|_{\gamma^s}^j \\
&\leq C_3\,|u-v|_{\gamma^s}\,|u_0|_{\gamma^s}^p.
\end{aligned}$$

Then we have

$$\begin{aligned}
\sup_{t\in[0,T]} |\Phi(u(t)) - \Phi(v(t))|_{\gamma^s} &\leq C\,|u_0|_{\gamma^s}^p \int_0^t \frac{|u-v|_{\gamma^s}}{(t-\tau)^{\frac{r}{2}}} d\tau \\
&\leq C\,|u_0|_{\gamma^s}^p\, T^{1-\frac{r}{2}} \sup_{t\in[0,T]} \left(|u-v|_{\gamma^s}\right).
\end{aligned}$$

The map Φ is strictly contracting if

$$T^{1-\frac{r}{2}} < \frac{1}{C\,|u_0|_{\gamma^s}^p}.$$

It remains to prove the continuity with respect to the initial data. Duhamel's formula gives for $t \in [0,T]$, $T^{1-\frac{r}{2}} \leq \frac{1}{C_0 M^p}$

$$\begin{aligned}
|u-v|_{\gamma^s} \leq |u_0 - v_0|_{\gamma^s} &+ \int_0^t |f(u) - f(v)|_{\gamma^s}\, d\tau \\
&|u_0 - v_0|_{\gamma^s} + C' T^{1-\frac{r}{2}} \left(\sum_{i+j=p} |u_0|_{\gamma^s}^i\,|v_0|_{\gamma^s}^j\right) |u-v|_{\gamma^s} \\
&|u_0 - v_0|_{\gamma^s} + C' T^{1-\frac{r}{2}} \left(\sum_{i+j=p} |u_0|_{\gamma^s}^i\,|v_0|_{\gamma^s}^j\right) \sup_{t\in[0,T]} \left(|u-v|_{\gamma^s}\right).
\end{aligned}$$

It involves

$$|u-v|_{\gamma^s} \leq C_1\,|u_0 - v_0|_{\gamma^s}.$$

$\square$

Remark 2.3. *Actually we can proove the local well-posedness for every γ using a parabolic regularisation*

$$u_t + u_x + u_{xxx} + u^p u_x + \mathscr{L}_\gamma(u) - \epsilon u_{xx} = 0.$$

Using lemma 2.1 with $\gamma(\xi) = \xi^2$, the same computations as theorem 2.2 and taking the limit $\epsilon \to 0$ give the result [Iór90, BS75].

3 Global well-posedness

We work here under the hypothesis of the local theorem and study the global well-posedness of the damped KdV equation. We use here an energy method [BS75, BS74]

Theorem 3.1. *If $p < 4$, for all γ, the unique solution is global in time, valued in $H^1(\mathbb{R})$. Else ($p \geq 4$), there exists a constant $\theta > 0$ such that if $\gamma(\xi) \geq \theta$, $\forall \xi \in \mathbb{R}$ then the unique solution is global in time, valued in $H^2(\mathbb{R})$.*

Proof. **Case $p < 4$:** We begin by introducing $N(u)$ and $E(u)$, two invariants of the KdV equation without the damping term, which are the L^2-norm and the energy. Their expressions are

$$\begin{aligned}
N(u) &= \int_{\mathbb{R}} u^2 dx = \|u\|_{L^2}^2 \,, \\
E(u) &= \frac{1}{2}\int_{\mathbb{R}} u_x^2 dx - \frac{1}{(p+1)(p+2)}\int_{\mathbb{R}} u^{p+2} dx = \frac{1}{2}\|u_x\|_{L^2}^2 - \frac{1}{(p+1)(p+2)}\|u\|_{L^{p+2}}^{p+2} .
\end{aligned}$$

We first multiply (1) by u and we integrate with respect to x. Then we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2 dx + \int_{\mathbb{R}} \mathscr{L}_\gamma(u) u dx = 0.$$

Integrating with respect to time, we obtain

$$\int_{\mathbb{R}} u^2 dx + 2\int_0^t \left(\int_{\mathbb{R}} \mathscr{L}_\gamma(u) u dx\right) d\tau = \int_{\mathbb{R}} u_0^2 dx.$$

Which can also be written as

$$N(u) + 2\int_0^t |u|_\gamma^2 \, d\tau = N(u_0).$$

We deduce from that expression that $N(u)$ is a decreasing function and $\int_0^t |u|_\gamma^2 \, d\tau$ is bounded independently of t by $N(u_0)$. Now, we multiply (1) by $u_{xx} + \frac{u^{p+1}}{p+1}$ and we integrate with respect to x. Then we have

$$\frac{d}{dt}\left(\int_{\mathbb{R}} -\frac{u_x^2}{2} + \frac{u^{p+2}}{(p+1)(p+2)} dx\right) - \int_{\mathbb{R}} \mathscr{L}_\gamma(u_x) u_x dx + \int_{\mathbb{R}} \mathscr{L}_\gamma(u)\left(\frac{u^{p+1}}{p+1}\right) dx = 0.$$

Integrating with respect to time, we obtain

$$E(u) + \int_0^t |u_x|_\gamma^2 \, d\tau - \int_0^t \left(\int_{\mathbb{R}} \mathscr{L}_\gamma(u)\left(\frac{u^{p+1}}{p+1}\right) dx\right) d\tau = E(u_0).$$

From this expression, we have

$$\begin{aligned}
\int_{\mathbb{R}} u_x^2 dx &= E(u) + \int_{\mathbb{R}} \frac{u^{p+2}}{(p+1)(p+2)} \\
&\leq E(u_0) + \int_{\mathbb{R}} \frac{u^{p+2}}{(p+1)(p+2)} + \int_0^t \left(\int_{\mathbb{R}} \mathscr{L}_\gamma(u)\left(\frac{u^{p+1}}{p+1}\right) dx\right) d\tau \\
&\leq E(u_0) + \frac{1}{(p+1)(p+2)}\|u\|_{L^\infty}^p \|u\|_{L^2}^2 + \left(\sup_{0\leq\tau\leq t}\|u\|_{L^\infty}^p\right)\frac{1}{p+1}\int_0^t \left(\int_{\mathbb{R}} \mathscr{L}_\gamma(u) u dx\right) d\tau.
\end{aligned}$$

Using the inequality $\|u\|_{L^\infty}^2 \leq 2\|u\|_{L^2}\|u_x\|_{L^2}$ and because $\int_{\mathbb{R}} |\mathscr{L}_\gamma(u)u| = \int_{\mathbb{R}} \mathscr{L}_\gamma(u)u$,

$$\int_{\mathbb{R}} u_x^2 dx \leq E(u_0) + \frac{2^{\frac{p}{2}}}{(p+1)(p+2)} \|u\|_{L^2}^{2+\frac{p}{2}} \|u_x\|_{L^2}^{\frac{p}{2}} + \sup_{0\leq\tau\leq t} \left(2^{\frac{p}{2}} \|u\|_{L^2}^{\frac{p}{2}} \|u_x\|_{L^2}^{\frac{p}{2}}\right) \int_0^t |u|_\gamma^2 \, d\tau.$$

Since $\|u\|_{L^2} \leq \|u_0\|_{L^2}$

$$\int_{\mathbb{R}} u_x^2 dx \leq C_0 + C_1 \|u_x\|_{L^2}^{\frac{p}{2}} + C_2 \sup_{0\leq\tau\leq t} \|u_x\|_{L^2}^{\frac{p}{2}}.$$

Then

$$\sup_{0\leq\tau\leq t} \|u_x\|_{L^2}^2 - C \sup_{0\leq\tau\leq t} \|u_x\|_{L^2}^{\frac{p}{2}} \leq C_0. \tag{3}$$

If there exists $T > 0$ such that $\lim_{t\to T} \|u_x\|_{L^2} = +\infty$ then $\|u_x\|_{L^2}^2 - C\|u_x\|_{L^2}^{\frac{p}{2}} \to +\infty$ since $p < 4$ and this is impossible because of (3). Consequently, $\|u_x\|_{L^2}$ is bounded for all t and so is the H^1-norm.

Case $p \geq 4$:

We estimate the L^2-norm of u_{xx}. We multiply (1) with u_{xxxx} and we integrate with respect to x. Then we have

$$\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}} u_{xx}^2 dx\right) + \int_{\mathbb{R}} \mathscr{L}(u) u_{xxxx} dx = -\int_{\mathbb{R}} u^p u_x u_{xxxx} dx. \tag{4}$$

Using two integrations by part, we have

$$\int_{\mathbb{R}} \mathscr{L}(u) u_{xxxx} dx = \int_{\mathbb{R}} \mathscr{L}(u_{xx}) u_{xx} dx.$$

Let us work on the last term. Using integrations by part, we have

$$-\int_{\mathbb{R}} u^p u_x u_{xxxx} dx = -\frac{5p}{2}\int_{\mathbb{R}} u^{p-1} u_x u_{xx}^2 dx - p(p-1)\int_{\mathbb{R}} u^{p-2} u_x^3 u_{xx} dx.$$

It follows that

$$-\int_{\mathbb{R}} u^p u_x u_{xxxx} dx \leq \frac{5p}{2}\|u\|_\infty^{p-1}\|u_x\|_\infty \|u_{xx}\|_{L^2}^2 + p(p-1)\|u\|_\infty^{p-2}\|u_x\|_\infty^2 \int_{\mathbb{R}} |u_x u_{xx}| dx.$$

But, from the Cauchy-Schwarz inequality

$$\int_{\mathbb{R}} |u_x u_{xx}| dx \leq \|u_x\|_{L^2}\|u_{xx}\|_{L^2}.$$

Then we have

$$-\int_{\mathbb{R}} u^p u_x u_{xxxx} dx \leq \frac{5p}{2}\|u\|_\infty^{p-1}\|u_x\|_\infty \|u_{xx}\|_{L^2}^2 + p(p-1)\|u\|_\infty^{p-2}\|u_x\|_\infty^2 \|u_x\|_{L^2}\|u_{xx}\|_{L^2}.$$

Using the inequalty $\|u\|_\infty^2 \leq 2\|u\|_{L^2}\|u_x\|_{L^2}$, we obtain

$$\begin{aligned}
-\int_{\mathbb{R}} u^p u_x u_{xxxx} dx \leq &\left[\frac{5p}{2}\left(2\|u\|_{L^2}^{\frac{3}{2}}\|u_{xx}\|_{L^2}^{\frac{1}{2}}\right)^{\frac{p-1}{2}} \|u\|_{L^2}^{\frac{1}{4}}\|u_x x\|_{L^2}^{\frac{3}{4}}\right. \\
&\left. + p(p-1)\left(2\|u\|_{L^2}^{\frac{3}{2}}\|u_{xx}\|_{L^2}^{\frac{1}{2}}\right)^{\frac{p-2}{2}} \|u\|_{L^2}\|u_{xx}\|_{L^2}\right] \|u_{xx}\|_{L^2}^2. \\
&=: \Omega\left(\|u\|_{L^2}, \|u_{xx}\|_{L^2}\right)\|u_{xx}\|_{L^2}^2.
\end{aligned}$$

From (4), it leads to the inequality

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|_{L^2} + \int_{\mathbb{R}} \mathscr{L}_\gamma(u_{xx})u_{xx} - \Omega u_{xx}^2 dx \le 0.$$

But

$$\begin{aligned}\int_{\mathbb{R}} \mathscr{L}_\gamma(u_{xx})u_{xx} - \Omega u_{xx}^2 dx &= \int_{\mathbb{R}} \left[\widehat{\mathscr{L}_\gamma(u_x x)}\overline{\widehat{u_{xx}}} - \Omega\widehat{u_{xx}}\overline{\widehat{u_{xx}}}\right] d\xi \\ &= \int_{\mathbb{R}} (\gamma(\xi) - \Omega)\,|\widehat{u_{xx}}|^2\, d\xi.\end{aligned}$$

The function Ω is increasing for its two arguments. We previously notice that $\|u(\cdot,t)\|_{L^2}$ is an decreasing function with respect to the time. Then, if $\gamma(\xi) - \Omega|_{t=0} \ge 0$, $\|u_{xx}(\cdot,t)\|_{L^2}$ does not increase for $t \ge 0$. Particularly, if $\gamma(\xi) \ge \Omega\left(\|u_0\|_{L^2}, \|u_{0xx}\|_{L_2}\right) =: \theta$, the semi-norm $\|u_{xx}(\cdot,t)\|_{L^2}$ is bounded by its values at $t = 0$. □

Remark 3.2. *This result is also true on the torus $\mathbb{T}(0,L)$ where the operator $\mathscr{L}_\gamma$ is defined by its Fourier symbol*

$$\widehat{\mathscr{L}_\gamma(u)}(k) := \gamma_k \hat{u}_k.$$

Here $\hat{u}_k$ is the $k-th$ Fourier coefficient of u and $(\gamma_k)_{k\in\mathbb{Z}}$ are positive real numbers chosen such that

$$\int_{\mathbb{T}} u(x)\mathscr{L}_\gamma(u)d\mu(x) = \sum_{k\in\mathbb{Z}} \gamma_k|\hat{u}_k|^2 \ge 0.$$

4 Numerical results

In this part, we illustrate the theorem 3.1 numerically. Our purpose is first to find similar results as in [BDKM96] i.e. find a γ_k constant such that the solution does not blow up. Then build a sequence of γ_k, still preventing the blow-up, such that $\lim_{|k|\to+\infty} \gamma_k = 0$. Since dKdV is a low frequencies problem, we do not need to damp all the frequencies.

4.1 Computation of the damping

In order to find the suitable damping, one may use the dichotomy. We remind that our goal is to prevent the blow-up, i.e., avoid that $\lim_{t\to+\infty} \|u\|_{H^1} = +\infty$. Let us begin finding a constant damping $\mathscr{L}_\gamma(u) = \gamma u$ as weak as possible. We mean by weak that γ has to be as lower as possible to prevents the blow up. Let γ_a respectively γ_e be the damping which prevents the explosion and which does not respectively. To initialize the dichotomy, we give a value to γ and we determine the initial values of γ_a and γ_e. Then from these two initial values, we bring them closer by using dichotomy. The method is detailed in algorithm 1 and illustrated in Figures 1 and 2.

Algorithm 1 γ_a and γ_e using dichotomy

Require: γ_0, ϵ
Initialisation of γ : $\gamma = \gamma_0$
Simulation with $\gamma_k = \gamma$
if Explosion **then**
  while Explosion **do**
    $\gamma = 2\gamma$
    Simulation with $\gamma_k = \gamma$
  end while
  $\gamma_a = \gamma$
  $\gamma_e = \frac{\gamma}{2}$
else
  while Damping **do**
    $\gamma = \frac{\gamma}{2}$
    Simulation with $\gamma_k = \gamma$
  end while
  $\gamma_e = \gamma$
  $\gamma_a = 2\gamma$
end if
while $|\gamma_a - \gamma_e| > \epsilon$ **do**
  $\gamma = \frac{\gamma_a + \gamma_e}{2}$
  Simulation with $\gamma_k = \gamma$
  if Explosion **then**
    $\gamma_e = \gamma$
  else
    $\gamma_a = \gamma$
  end if
end while

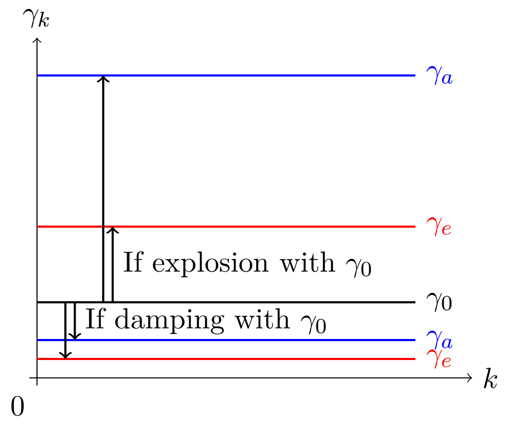

Figure 1: Initialization

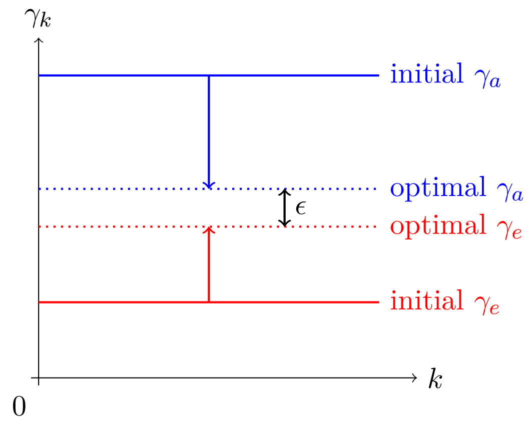

Figure 2: Dichotomy

W extend the method to frequencies bands in order to build sequencies γ_k decreasing with respect to $|k|$ and tending to 0 when $|k|$ tends to the infinity. So we begin by defining the frequencies bands ($N_1 < N_2 < \ldots$) and we proceed as previously but only on the frequencies $|k| \geq N_i$. The method is described in algorithm 2 and illustrated in Figure 3 and 4.

Algorithm 2 γ_a and γ_e on the band using dichotomy

Require: γ_a, N and Nb_iter
Initialisation $\gamma = \gamma_a$
$\gamma_{|k|>N} = 0$
Simulation with γ
if Damping **then**
  return $\gamma_a = \gamma$
else
  $\gamma = \gamma_a$
  while Damping **do**
    $\gamma_{|k|>N} = \frac{\gamma_{|k|>N}}{2}$
    Simulation with γ
  end while
  $\gamma_e = \gamma$
  $\gamma_{a,|k|>N} = 2\gamma_{|k|>N}$
end if
for $i = 1$ to Nb_iter **do**
  $\gamma_{|k|>N} = \frac{\gamma_{a,|k|>N} + \gamma_{e,|k|>N}}{2}$
  Simulation with γ
  if Explosion **then**
    $\gamma_{e,|k|>N} = \gamma_{|k|>N}$
  else
    $\gamma_{a,|k|>N} = \gamma_{|k|>N}$
  end if
end for

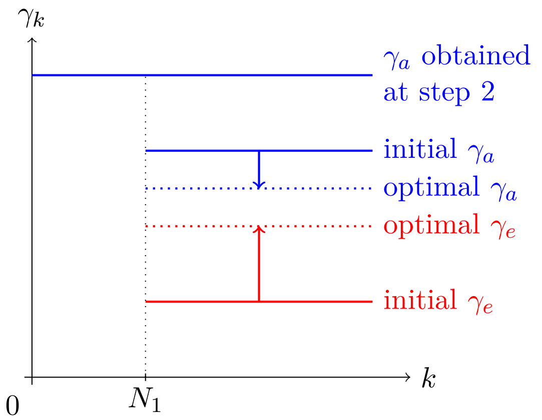

Figure 3: Initialization

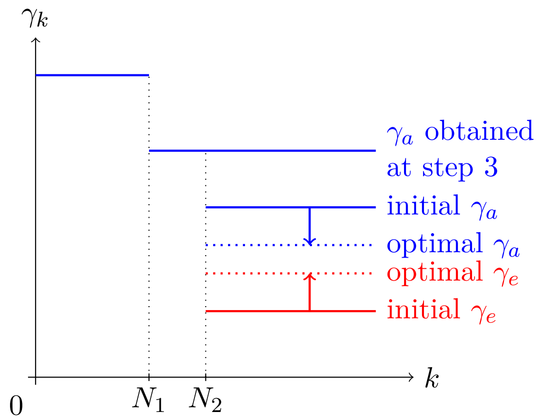

Figure 4: Find the damping

4.2 Numerical scheme

Numerous schemes were introduced in [CheSad]. Here we chose a Sanz-Serna scheme for the discretisation in time. In space, we use the FFT. Actually, the scheme is written, for all k, as

$$\left(1+\frac{\Delta t}{2}(ik-ik^3+\gamma_k)\right)\widehat{u^{(n+1)}}(k)=\left(1-\frac{\Delta t}{2}(ik-ik^3+\gamma_k)\right)\widehat{u^{(n)}}(k)\\ -\frac{ik\Delta t}{p+1}\mathscr{F}\left[\left(\frac{u^{(n+1)}+u^{(n)}}{2}\right)^{p+1}\right](k).$$

We find $\widehat{u^{(n+1)}}_k$ with a fixed-point method. In order to have a good look of the blow-up, we also use an adaptative time step.

4.3 Simulations

We consider the domain $[-L,L]$ where $L=50$. We take as initial datum a disturbed soliton, written as

$$u_0(x)=1.01\times\left(\frac{(p+1)(p+2)(c-1)}{2}\right)^{\frac{1}{p}}\cosh^{-\frac{2}{p}}\left(\pm\sqrt{\frac{p(c-1)}{4}}(x-ct-d)\right),$$

where $p=5$, $c=1.5$ and $d=0.2L$. We discretise the space in 2^{11} points. The Figure 5 shows the solution whithout damping, i.e., $\gamma_k=0$, $\forall k$. We observe that the L^2-norm of u_x increases strongly and the solution tends to a wavefront (as in [BDKM96]).

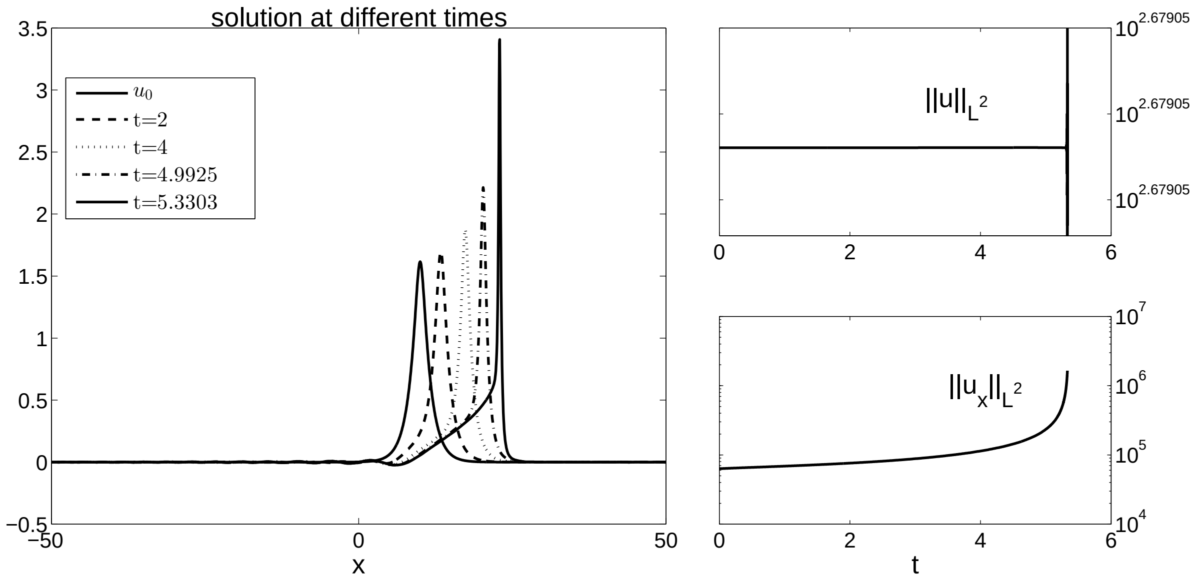

Figure 5: At left, solution at different times $t = 0$, 2, 4, 4.9925 and 5.3303. At right, H^1-norm and L^2-norm evolution without damping and a perturbed soliton as initial datum. Here $p = 5$.

Using the methods introduced previously, we first find two optimal constant dampings $\gamma_e = 0.0025$ and $\gamma_a = 0.0027$. As we can see in Figure 6, γ_e does not prevent the blow up. In the opposite in Figure 7 γ_a does. And we also notice that the two dampings are quite close.

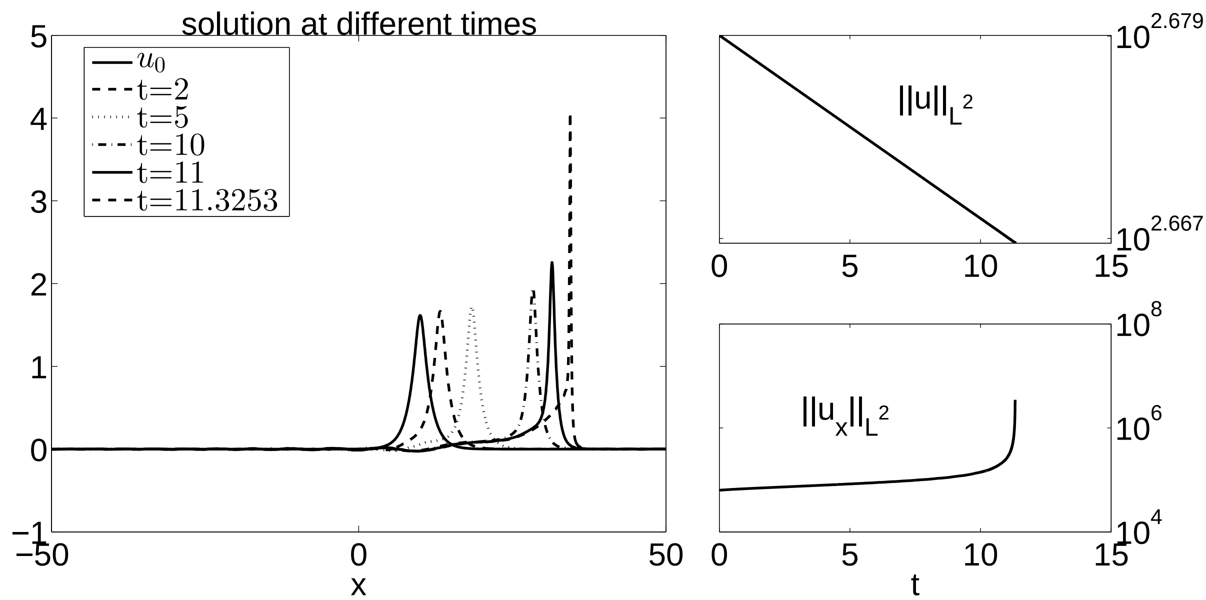

Figure 6: At left, solution at different times $t = 0$, 2, 5, 10, 11 and 11.3253. At right, H^1-norm and L^2-norm evolution with $\gamma_k = 0.0025$ and a perturbed soliton as initial datum. Here $p = 5$.

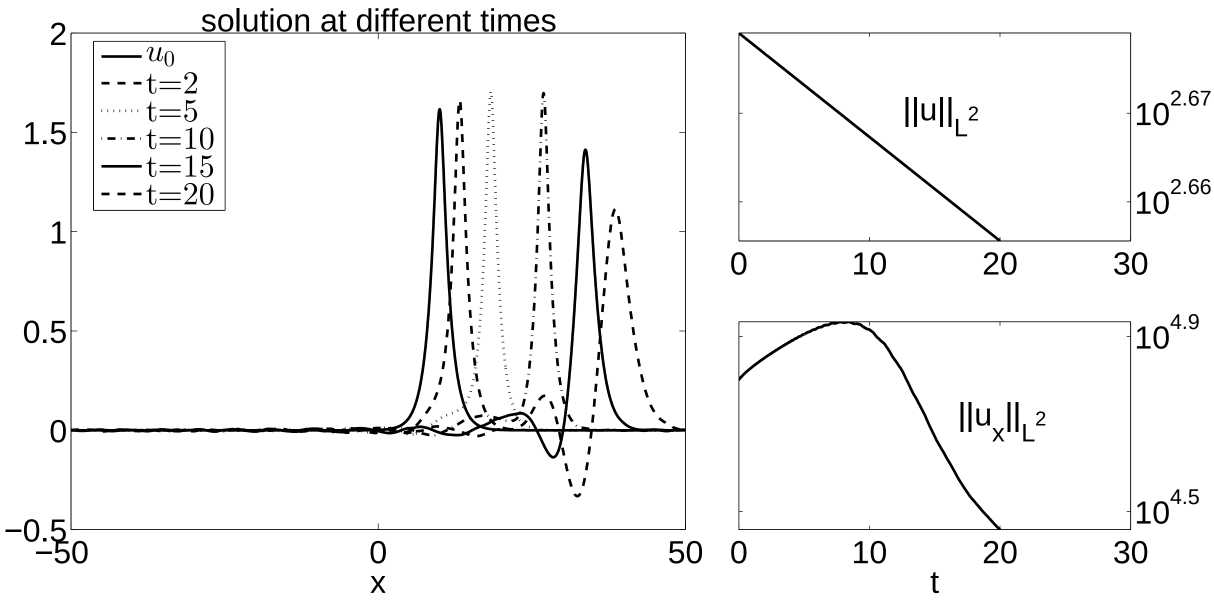

Figure 7: At left, solution at different times $t = 0$, 2, 5, 10, 15 and 20. At right, H^1-norm and L^2-norm evolution with $\gamma_k = 0.0027$ and a perturbed soliton as initial datum. Here $p = 5$.

Considering more general sequences, particularly such that $\lim_{|k|\to+\infty} \gamma_k = 0$. Using algorithm 2, Figure 8 shows that the sequence (γ_a) as a frontier between the dampings which prevent the blow up and the other which do not. To illustrate this, we take two dampings written as gaussians. The first (denoted by γ_1) is build to be always above the sequence γ_a and the second (denoted by γ_2) to be always below. In Figures 9 and 10 we observe the damping $\gamma = \gamma_1$ prevents the blow up. But if we take $\gamma = \gamma_2$, the solution blows-up.

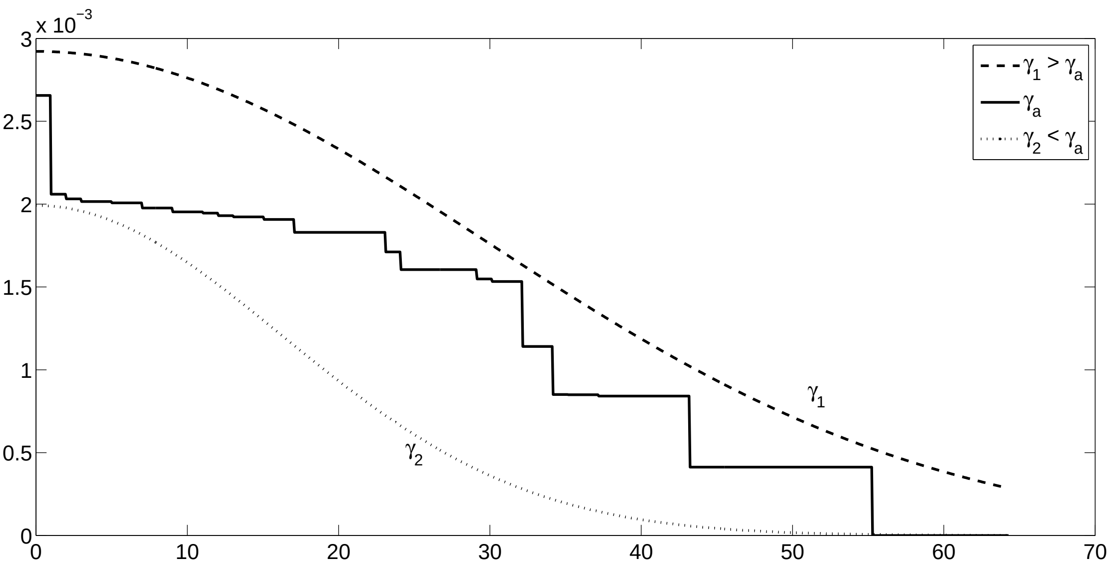

Figure 8: Example of a build damping. Here the initial datum is the perturbed soliton. Here $p = 5$.

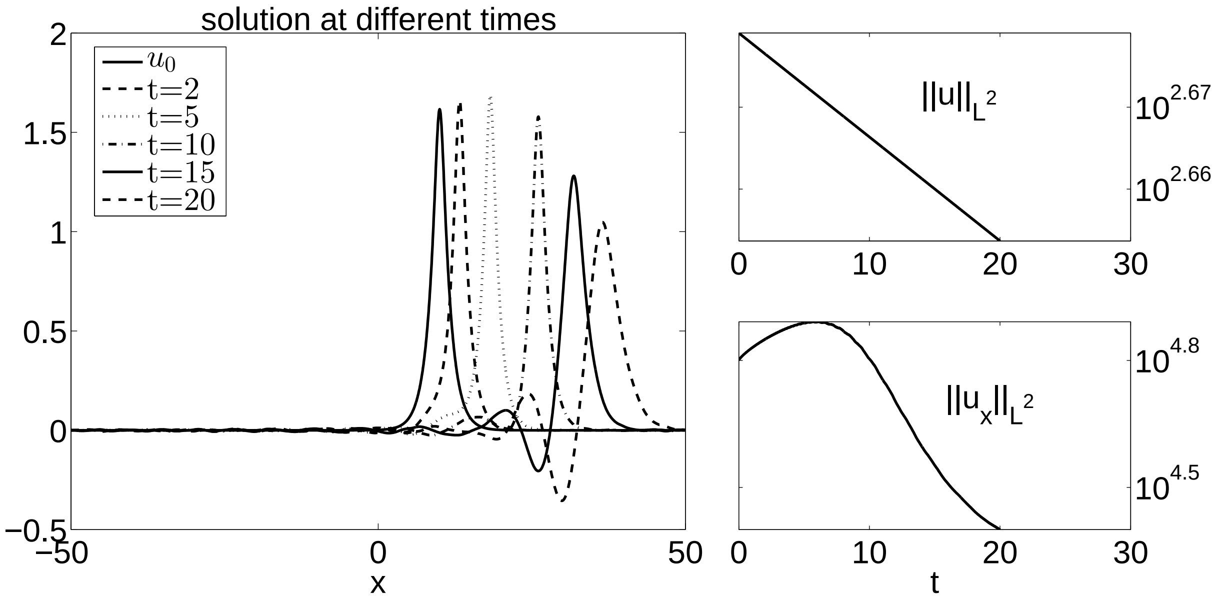

Figure 9: At left, solution at different times $t = 0$, 2, 5, 10, 15 and 20. At right, H^1-norm and L^2-norm evolution with $\gamma = \gamma_1$ and a perturbed soliton as initial datum. Here $p = 5$.

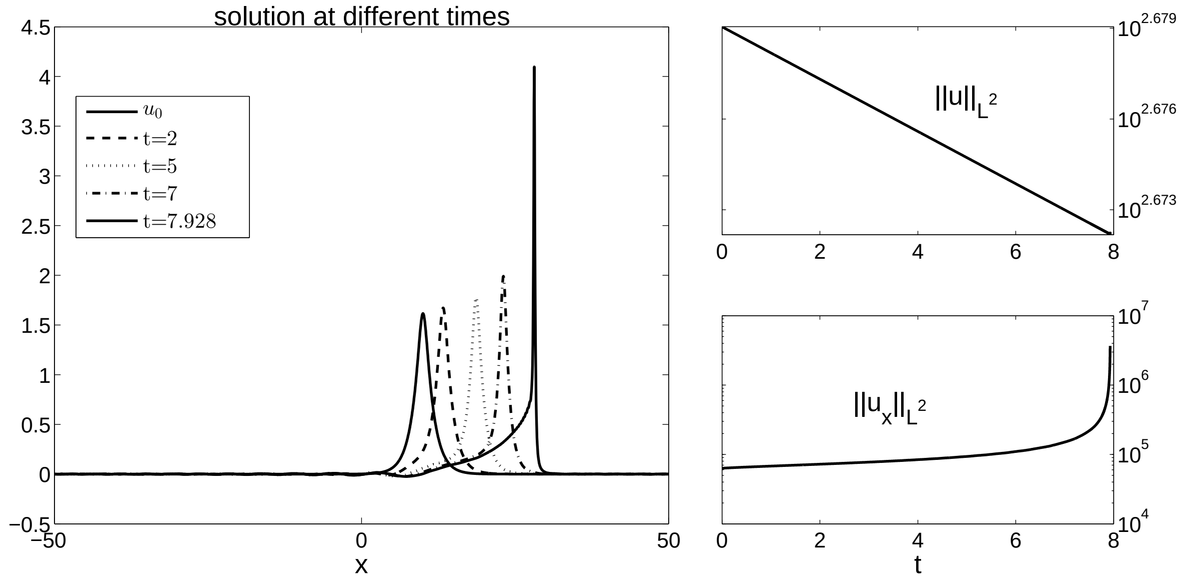

Figure 10: At left, solution at different times $t = 0$, 2, 5, 7 and 7.928. At right, H^1-norm and L^2-norm evolution with $\gamma = \gamma_2$ and a perturbed soliton as initial datum. Here $p = 5$.

Conclusion

We studied the behavior of the damped generalized KdV equation. If $p < 4$, the solution does not blow-up whereas if $p \geq 4$, it can. To prevent the blow-up, the term γ defining the damping has to be large enough. In particular, we build a sequence of γ which vanishes for high frequencies. This frequential approach for the damping seems useful for low frequencies problem.

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